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Scalar Field Theories On The World Sheet: Cutoff Independent Treatment ¹

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Abstract

Following earlier work on the same topic, we consider once more scalar field theories on the world sheet parametrized by the light cone coordinates. For most of the way, we use the same approach as in the previous work, but there is an important new development. To avoid the light cone singularity at $p^+ = 0$, one world sheet coordinate had to be discretized, introducing a cutoff into the model. In the earlier work, this cutoff could not be removed, making the model unreliable. In the present article, we show that, by a careful choice of the mass counter term, both the infrared singularity at $p^+ = 0$ and the ultraviolet mass divergences can be simultaneously eliminated. We therefore finally have a cutoff independent model on a continuously parametrized world sheet. We study this model in the mean field approximation, and as

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before, we find solitonic solutions. Quantizing the solitonic collective coordinates gives rise to a string like model. However, in contrast to the standard string model, the trajectories here are not in general linear but curved.

1. Introduction

The present article is likely the last in a series of several articles, starting with [1]. The idea behind this work was to sum the planar graphs of field theory on a world sheet parametrized by the light cone variables, based on 't Hooft's pioneering paper [2]. The original field theory studied in this approach was a scalar with ϕ^3 interaction, and this was later generalized to more complicated and more interesting models [3, 4]. This paper is a follow up to reference [5], using mostly the same approach developed in previous publications. The models under consideration are scalar theories with both only ϕ^3 interaction and also with a mixture of ϕ^4 added. In [5], using both mean field and variational approximations on the world sheet, solitonic classical solutions on the world sheet were constructed, and a certain set of quantum fluctuations about the solitonic solutions were shown to have a string like spectrum.

These computations suffer from two kinds of divergences: One of them is the field theoretic ultraviolet divergences, which are eliminated by the standard renormalization procedure. The second one is an infrared divergence due to the choice of the light cone coordinates. In the previous work, this infrared problem was temporarily circumvented by the discretizing the σ coordinate on the world sheet in steps of a , but then, several quantities of physical interest, such as the ground state energy, diverged as $1/a^2$ in the limit $a \rightarrow 0$. This casts serious doubt on the whole approach; in a massive theory that we are dealing with, since there is no infrared problem, this singularity should be spurious [6]. This also prevents us from discussing the continuum limit on the world sheet, which is after all the model of interest.

The main result of this paper is that this singularity is indeed spurious, and it can be eliminated by a mass counter term. It is surprising and highly satisfying that the same counter terms that are needed to cancel the ultraviolet mass divergences also automatically cancel the infrared singularity at $a = 0$, a possibility which was overlooked in [5]. As we shall see later in section 4, the key to the elimination of this singularity is in taking the limit $a \rightarrow 0$ with proper care.

With the singularity at $a = 0$ eliminated, there is no obstacle to taking the continuum limit, and to applying the mean field approximation to the continuum model. Just as in [5], we construct the solitonic solution and with the soliton as background, we study the quantum fluctuations. The solitonic solutions are of interest because they describe a non-perturbative

feature of field theory. Also, as we shall see later, the soliton emerges from the summation of a dense set of graphs on the world sheet, which can be thought of as the condensation of these graphs. The existence of such a condensate on the world sheet is naturally expected to lead to a string description, an old idea that motivated some of the early work on this subject [7, 8].

To make the present work self contained and comprehensible, sections 2 and 3, as well as the early part of 4, are mostly devoted to the review of the earlier work, especially of reference [5]. In section 2, the world sheet picture of the planar graphs of the ϕ^3 field theory is reviewed, and in section 3, we describe the world sheet field theory, developed in [9], which reproduces these graphs. This theory is constructed in terms of a complex scalar field and a two component fermion field. In its formulation, a central role is played by the field ρ (eq.(4)), a composite of the fermions, which measures the density of the graphs on the world sheet.

In section 4, we discuss in detail the construction of the classical solitonic solutions of the world sheet field equations. These solutions are hybrid in character: The complex scalar ϕ is treated classically, but the fermions are still operators that satisfy the usual anti commutation relations. The reason for this hybridization is that there are certain overlap relations between interaction vertices (eqs. (20) and (21)), similar to those encountered in string theory, which we would like to treat exactly at this stage of the computation. These overlap relations are crucial for the successful renormalization of the model and the elimination of the singularity at $a = 0$. They follow from the algebra satisfied by the bilinears of the fermions (eq.(19)), and to keep this algebra intact, we solve the equations of motion by means of a “hybrid” ansatz (eq.(23)). This ansatz is then used to compute the corresponding Hamiltonian (eq.(25)), and the result is simplified as much as possible with the help of the overlap relations. We show that in this final form of the Hamiltonian, the ultraviolet mass divergences that appear in dimensions $D = 2, 4$ can be canceled by a mass counter term.

So far, the sigma coordinate is still discretized; in fact, without such a discretization, the overlap relations would be difficult to write down without encountering singular expressions. In section 5, we show that after scaling the scalar and fermionic fields by \sqrt{a} , the continuum limit $a \rightarrow 0$ can be taken directly without any difficulty. The result (eq.(30)) is free of both infrared and ultraviolet singularities, except for a log singularity in the coupling constant at $D = 4$. This singularity can also be circumvented by coupling constant renormalization. It is remarkable that the same mass term that

cancels the ultraviolet divergence also eliminates the infrared singularity.

The continuum limit comes with an additional bonus: The model is now invariant under the subgroup of Lorentz transformations that preserve the light cone, including the boost K_1 along the special direction 1. Invariance under this boost, broken when the sigma coordinate is discretized, is restored in the continuum limit. We hasten to add that, a priori, there is nothing wrong with this discretization, which amounts to the compactification of the light cone coordinate x^- [10, 11]. It is just that one can now decompactify without encountering any obstacles.

Now that we have a satisfactory model, free of divergences, we try approximations to make it tractable. In section 6, we introduce the mean field method, which was frequently used in the earlier work. In this approximation, the fields are replaced by their ground state expectation values, which are assumed to be independent of the coordinates σ and τ . This strategy is familiar from field theory. To find the ground state of the model in this approximation, we have to solve the classical equations of motion (eq.(36)). Although these equations can be investigated in all generality, in order to get a simple result in closed form, we consider only the limit of large ρ , or a dense set of graphs on the world sheet. As we have mentioned earlier, this limit is interesting from the point of view of solitonic solutions and string formation. In transverse dimensions $D = 1$ and $D = 2$, we get lower bounds on the coupling constants (eqs.(40)), necessary for the formation of the solitonic condensate on the world sheet. It is possible that the critical values of the coupling constants mark a phase transition from the weak coupling perturbative phase to the strong coupling condensate phase. We also compute the masses of the ground states in various dimensions, and at least in this approximation, there is no tachyonic instabilities.

In section 7, we consider an interaction which is an admixture of ϕ^3 and ϕ^4 for $D = 1$, to see whether the simultaneous cancellation of the ultraviolet and the infrared singularities by a mass counter term still goes through. It is highly satisfactory that this cancellation still works as before, showing that it is not a special feature of a pure ϕ^3 interaction.

Having constructed the solitonic solutions, in section 8, we study the quantum fluctuations in the solitonic background. Here, we focus exclusively on a particular set of fluctuations, which were also studied in the previous work [5]. They come about because the soliton, having a definite location, breaks the translation symmetry of the model (eq.(17)). It is then the standard procedure to introduce collective coordinates corresponding to transla-

tions. Upon quantization, these collective modes restore the spontaneously broken translation symmetry. They can therefore be identified as the Goldstone modes, and are expected to dominate the low energy regime. Also, as in the earlier work, we expect them to play an important role in string formation on the world sheet. As an additional bonus, in the mean field approximation, their contribution can be computed in closed form (eqs.(52) and (59)), and the result is a generalized free field theory.

In section 9, we determine the spectrum of the model, and find that, unlike the conventional string theory, the Regge trajectories are no longer linear. Only in the asymptotic limit when the density of graphs, ρ , tends to infinity, the standard string model with linear trajectories is recovered.

We end this section by comparing the present paper with the previous work, especially [5]. The main new result is that we can let the lattice spacing a go to zero, without encountering any singularities. This makes it possible to study the model on a continuous world sheet. In contrast, in the previous work, strictly speaking, the limit $a \rightarrow 0$ did not exist. We also have some new technical results, such as masses of the ground state excitations, and a new string like model with curved trajectories. But the main advance is in reliability; finally, there is a treatment of the scalar field theory on a world sheet free of cutoffs. Going back to the first sentence of this section, now that the model is free of obvious problems, we feel that the original project has reached a natural end. Of course, this remark only applies to scalar theories and the light cone setup. More physical theories with spin and Lorentz invariance [3, 4, 12, 13] remain as important problems for future research.

2. The World Sheet Picture

The planar graphs of ϕ^3 can be represented [2] on a world sheet parametrized by the light cone coordinates $\tau = x^+$ and $\sigma = p^+$ as a collection of horizontal solid lines (Fig.1), where the n 'th line carries a D dimensional transverse momentum \mathbf{q}_n . Two adjacent solid lines labeled by n and $n+1$ correspond to the light cone propagator

$$\Delta(\mathbf{p}_n) = \frac{\theta(\tau)}{2p^+} \exp \left(-i\tau \frac{\mathbf{p}_n^2 + m^2}{2p^+} \right), \quad (1)$$

where $\mathbf{p}_n = \mathbf{q}_n - \mathbf{q}_{n+1}$ is the momentum flowing through the propagator. A factor of the coupling constant g is inserted at the beginning and at the

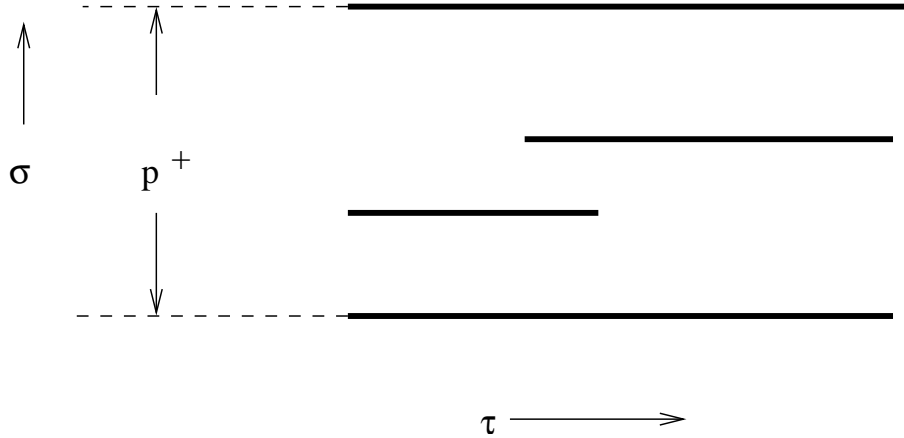


Figure 1: A Typical Graph

end of each line, where the interaction takes place. Ultimately, one has to integrate over all possible locations and lengths of the solid lines, as well as over the momenta they carry.

The propagator (1) is singular at $p^+ = 0$. It is well known that this is a spurious singularity peculiar to the light cone picture. To avoid this singularity temporarily, it is convenient to discretize the σ coordinate in steps of length a . A useful way of visualizing the discretized world sheet is pictured in Fig.2. The boundaries of the propagators are marked by solid lines as before, and the bulk is filled by dotted lines spaced at a distance a . For convenience, the σ coordinate is compactified by imposing periodic boundary conditions at $\sigma = 0$ and $\sigma = p^+$. In contrast, the boundary conditions at $\tau = \pm\infty$ are left arbitrary. We stress that, in a massive and hence infrared finite theory, the $p^+ = 0$ singularity should be absent. We will show how to eliminate it in the following sections, and this will allow us to go from a discrete to a continuous world sheet.

3. The World Sheet Field Theory

It was shown in [9] that the light cone graphs described above are reproduced by a world sheet field theory, which we now briefly review. We introduce the complex scalar field $\phi(\sigma, \tau, \mathbf{q})$ and its conjugate ϕ^\dagger , which at time τ annihilate (create) a solid line with coordinate σ carrying momentum

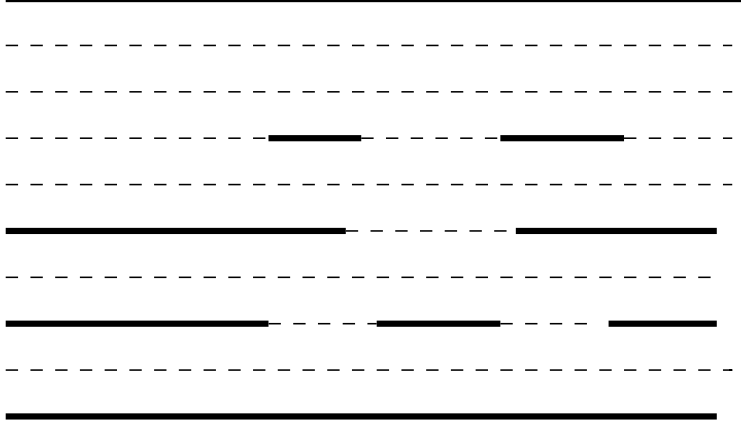


Figure 2: Solid And Dotted Lines

q. They satisfy the usual commutation relations

$$[\phi(\sigma, \tau, \mathbf{q}), \phi^\dagger(\sigma', \tau, \mathbf{q}')] = \delta_{\sigma, \sigma'} \delta(\mathbf{q} - \mathbf{q}'). \quad (2)$$

The vacuum, annihilated by the ϕ 's, represents the empty world sheet.

In addition, we introduce a two component fermion field $\psi_i(\sigma, \tau)$, $i = 1, 2$, and its adjoint $\bar{\psi}_i$, which satisfy the standard anti commutation relations. The fermion with $i = 1$ is associated with the dotted lines and $i = 2$ with the solid lines. The fermions are needed to avoid unwanted configurations on the world sheet. For example, multiple solid lines generated by the repeated application of ϕ^\dagger at the same σ would lead to over counting of the graphs. These redundant states can be eliminated by imposing the constraint

$$\int d\mathbf{q} \phi^\dagger(\sigma, \tau, \mathbf{q}) \phi(\sigma, \tau, \mathbf{q}) = \rho(\sigma, \tau), \quad (3)$$

where

$$\rho = \bar{\psi}_2 \psi_2, \quad (4)$$

which is equal to one on solid lines and zero on dotted lines. This constraint ensures that there is at most one solid line at each site.

Fermions are also needed to avoid another set of unwanted configurations. Propagators are assigned to adjacent solid lines and not to non-adjacent ones. To enforce this condition, it is convenient to define,

$$\mathcal{E}(\sigma_i, \sigma_j) = \prod_{k=i+1}^{k=j-1} (1 - \rho(\sigma_k)), \quad (5)$$

for $\sigma_j > \sigma_i$, and zero for $\sigma_j < \sigma_i$. The crucial property of this function is that it acts as a projection: It is equal to one when the two lines at σ_i and σ_j are separated only by the dotted lines; otherwise, it is zero. With the help of \mathcal{E} , the free Hamiltonian can be written as

$$\begin{aligned}
H_0 &= \frac{1}{2} \sum_{\sigma, \sigma'} \int d\mathbf{q} \int d\mathbf{q}' \frac{\mathcal{E}(\sigma, \sigma')}{\sigma' - \sigma} ((\mathbf{q} - \mathbf{q}')^2 + m^2) \\
&\times \phi^\dagger(\sigma, \mathbf{q}) \phi(\sigma, \mathbf{q}) \phi^\dagger(\sigma', \mathbf{q}') \phi(\sigma', \mathbf{q}') \\
&+ \sum_{\sigma} \lambda(\sigma) \left(\int d\mathbf{q} \phi^\dagger(\sigma, \mathbf{q}) \phi(\sigma, \mathbf{q}) - \rho(\sigma) \right), \tag{6}
\end{aligned}$$

where λ is a Lagrange multiplier enforcing the constraint (3). The evolution operator $\exp(-i\tau H_0)$, applied to states, generates a collection of free propagators, without, however, the prefactor $1/(2p^+)$.

One can also think of the Lagrange multiplier $\lambda(\sigma, \tau)$ as an Abelian gauge field on the world sheet. The corresponding gauge transformations are [14]

$$\begin{aligned}
\psi &\rightarrow \exp\left(-\frac{i}{2}\alpha\sigma_3\right)\psi, \quad \bar{\psi} \rightarrow \bar{\psi} \exp\left(\frac{i}{2}\alpha\sigma_3\right), \\
\phi &\rightarrow \exp(-i\alpha)\phi, \quad \phi^\dagger \rightarrow \exp(i\alpha)\phi^\dagger, \\
\lambda &\rightarrow \lambda - \partial_\tau\alpha. \tag{7}
\end{aligned}$$

This gauge invariance comes about because constraint (3) is time independent. Using the equations of motion,

$$\partial_\tau \left(\int d\mathbf{q} (\phi^\dagger \phi) - \rho \right) = 0,$$

and therefore the constraint is really needed only at a fixed τ , say, as an initial condition. This can be implemented by gauge fixing by requiring λ to be independent of the time τ ,

$$\lambda(\sigma, \tau) \rightarrow \lambda(\sigma),$$

by a suitable choice of gauge parameter α . In this time independent form, λ is analogous to the chemical potential in statistical mechanics. After gauge fixing λ , we have to impose the equation of motion with respect to it, or, equivalently, the constraint (3) on the states.

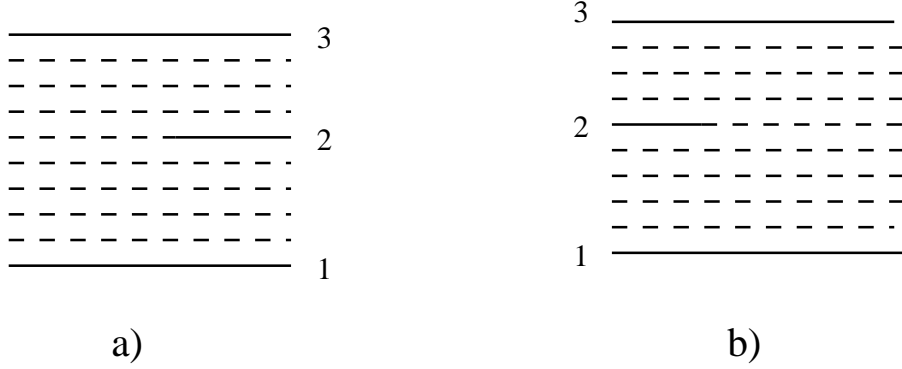


Figure 3: The Two ϕ^3 Vertices

Using the constraint, the free Hamiltonian can be written in a form more convenient for later application:

$$\begin{aligned}
H_0 = & \frac{1}{2} \sum_{\sigma, \sigma'} G(\sigma, \sigma') \left(\frac{1}{2} m_0^2 \rho(\sigma) \rho(\sigma') + \rho(\sigma') \int d\mathbf{q} (\mathbf{q}^2 + \mu^2) \phi^\dagger(\sigma, \mathbf{q}) \phi(\sigma, \mathbf{q}) \right. \\
& - \int d\mathbf{q} \int d\mathbf{q}' (\mathbf{q} \cdot \mathbf{q}') \phi^\dagger(\sigma, \mathbf{q}) \phi(\sigma, \mathbf{q}) \phi^\dagger(\sigma', \mathbf{q}') \phi(\sigma', \mathbf{q}') \Big) \\
& + \sum_{\sigma} \lambda(\sigma) \left(\int d\mathbf{q} \phi^\dagger(\sigma, \mathbf{q}) \phi(\sigma, \mathbf{q}) - \rho(\sigma) \right), \tag{8}
\end{aligned}$$

where we have defined

$$G(\sigma, \sigma') = \frac{\mathcal{E}(\sigma, \sigma') + \mathcal{E}(\sigma', \sigma)}{|\sigma - \sigma'|}. \tag{9}$$

There is a redundancy in the above equation: the mass is split into two pieces according to

$$m^2 = m_0^2 + 2\mu^2.$$

This redundancy will prove useful later on.

Next, we introduce the interaction term. Two kinds of interaction vertices, corresponding to ϕ^\dagger creating a solid line or ϕ destroying a solid line, are pictured in Fig.3.

We take care of the prefactors of the form $1/(p^+)$ in (1) by attaching a factor of

$$\frac{1}{\sqrt{p_{12}^+ p_{23}^+ p_{13}^+}} = \frac{1}{\sqrt{(\sigma_2 - \sigma_1)(\sigma_3 - \sigma_2)(\sigma_3 - \sigma_1)}} \tag{10}$$

to each vertex. The interaction term in the Hamiltonian can now be written as

$$H_I = g\sqrt{a} \sum_{\sigma} \int d\mathbf{q} \left(\mathcal{V}(\sigma) \rho_+(\sigma) \phi(\sigma, \mathbf{q}) + \rho_-(\sigma) \mathcal{V}(\sigma) \phi^\dagger(\sigma, \mathbf{q}) \right), \quad (11)$$

where g is the coupling constant,

$$\mathcal{V}(\sigma) = \sum_{\sigma_1 < \sigma} \sum_{\sigma < \sigma_2} \frac{W(\sigma_1, \sigma_2)}{\sqrt{(\sigma - \sigma_1)(\sigma_2 - \sigma_1)(\sigma_2 - \sigma)}}, \quad (12)$$

where,

$$W(\sigma_1, \sigma_2) = \rho(\sigma_1) \mathcal{E}(\sigma_1, \sigma_2) \rho(\sigma_2). \quad (13)$$

and

$$\rho_+ = \bar{\psi}_1 \psi_2, \quad \rho_- = \bar{\psi}_2 \psi_1. \quad (14)$$

Here is a brief explanation of the origin of various terms in H_I : The factors ρ_{\pm} are there to pair a solid line with an $i = 2$ fermion and a dotted line with an $i = 1$ fermion. The factor of \mathcal{V} ensures that the pair of solid lines 12 and 23 in Fig.3 are separated by only dotted lines, without any intervening solid lines. Apart from an overall factor, the vertex defined above is very similar to the bosonic string interaction vertex in the light cone picture. Taking advantage of the properties of \mathcal{E} discussed following eq.(5), we have written an explicit representation of this overlap vertex. Finally, the factor of \sqrt{a} multiplying the coupling constant is needed to convert sums over σ into integrals in the continuum limit.

The total Hamiltonian is given by

$$H = H_0 + H_I \quad (15)$$

and the corresponding action by

$$S = \int d\tau \left(\sum_{\sigma} \left(i\bar{\psi} \partial_{\tau} \psi + i \int d\mathbf{q} \phi^\dagger \partial_{\tau} \phi \right) - H(\tau) \right). \quad (16)$$

For later use, we note that the theory is invariant under

$$\phi(\sigma, \tau, \mathbf{q}) \rightarrow \phi(\sigma, \tau, \mathbf{q} + \mathbf{r}), \quad (17)$$

where \mathbf{r} is a constant vector.

4. Classical Solutions And Mass Renormalization

In this section, we look for classical solutions to the equations of motion resulting from the action (16). However, it was pointed out in [5] that treating the ρ 's as classical fields is problematic. It implies factorization of the expectation values of the products of the ρ 's. For example,

$$\langle \rho^2 \rangle \rightarrow \langle \rho \rangle \langle \rho \rangle = \rho_0^2, \quad (18)$$

and similarly for higher products. On the other hand, treated exactly, ρ takes on only the discrete values 0 and 1, and satisfies the identities

$$\rho^2(\sigma) = \rho(\sigma), \quad \rho_+(\sigma)\rho_-(\sigma) = 1 - \rho(\sigma), \quad \rho_-(\sigma)\rho_+(\sigma) = \rho(\sigma). \quad (19)$$

From these, one can derive two further identities

$$\begin{aligned} G(\sigma, \sigma') \rho(\sigma') \rho_-(\sigma) W(\sigma_1, \sigma_2) &= \\ \left(\delta_{\sigma', \sigma_2} \frac{1}{\sigma_2 - \sigma} + \delta_{\sigma', \sigma_1} \frac{1}{\sigma - \sigma_1} \right) \rho_-(\sigma) W(\sigma_1, \sigma_2), \end{aligned} \quad (20)$$

and

$$W(\sigma_1, \sigma_2) \rho_+(\sigma) \rho_-(\sigma) W(\sigma'_1, \sigma'_2) = \delta_{\sigma_1, \sigma'_1} \delta_{\sigma_2, \sigma'_2} W(\sigma_1, \sigma_2). \quad (21)$$

One can also understand eqs.(20, 21) geometrically from the overlap properties of the vertices in Fig.3. Apart from an overall factor, these are structurally the same as the corresponding string vertices, and in particular, they satisfy the same overlap relations. These overlap relations turn out to be crucial for the elimination of both ultraviolet divergences and the singularity at $a = 0$, which is the reflection of the original $p^+ = 0$ singularity in the propagator (1). If present, this singularity would prevent us from taking the continuum limit of the model.

The factorization ansatz (18) violates the identities (19) and consequently the overlap relations. To overcome this problem, we treat the ϕ 's as classical fields, but keep the ρ 's as operators satisfying eqs.(19) in the intermediate stages of the computation. The strategy is first to simplify the expressions as much as possible using the overlap relations (20, 21) before making any approximations.

We will now search for solutions $\phi_0(\sigma, \mathbf{q})$ that are time independent (solitonic) and whose dependence on \mathbf{q} is rotationally invariant. The equation

motion for ϕ_0 then simplifies to

$$\left(2\lambda(\sigma) + \sum_{\sigma'} G(\sigma, \sigma') \rho(\sigma') (\mathbf{q}^2 + \mu^2) \right) \phi_0(\sigma, \mathbf{q}) = -2g\sqrt{a} \rho_-(\sigma) \mathcal{V}(\sigma). \quad (22)$$

To solve this equation, we make the following ansatz for ϕ_0 :

$$\phi_0(\sigma, \mathbf{q}) = \sum_{\sigma_1 < \sigma} \sum_{\sigma < \sigma_2} \rho_-(\sigma) W(\sigma_1, \sigma_2) \tilde{\phi}_0(\sigma, \sigma_1, \sigma_2, \mathbf{q}). \quad (23)$$

where $\tilde{\phi}$ is a c-number and all the operator dependence is in $\rho_- W$. The motivation for this ansatz is that the operators appearing on either side of the equation are the same; namely $\rho_-(\sigma) W(\sigma_1, \sigma_2)$. Only the multiplying c-numbers are different. To show this for the left hand side of eq.(22), we reduce the operator product

$$G(\sigma, \sigma') \rho(\sigma') W(\sigma_1, \sigma_2)$$

to the form $\rho_-(\sigma) W(\sigma_1, \sigma_2)$ using eq.(20). The right hand side of (22) is also proportional to $\rho_-(\sigma) W(\sigma_1, \sigma_2)$ because of the form of \mathcal{V} (eq.(12)). We can then solve this equation by equating the c-number coefficient of the operator $\rho_- W$ on both sides. The result is

$$\begin{aligned} \tilde{\phi}_0(\sigma, \sigma_1, \sigma_2, \mathbf{q}) &= -\frac{2g\sqrt{a}}{\left(2\lambda(\sigma) + (\mathbf{q}^2 + \mu^2) \left(\frac{\sigma_2 - \sigma_1}{(\sigma_2 - \sigma)(\sigma - \sigma_1)} \right) \right)} \\ &\times \frac{1}{\sqrt{(\sigma - \sigma_1)(\sigma_2 - \sigma_1)(\sigma_2 - \sigma)}}, \end{aligned} \quad (24)$$

and the solution for ϕ_0^\dagger is given by the Hermitian conjugate expression.

Next, we define H_c by replacing ϕ by the above ϕ_0 in the Hamiltonian,

$$H_c = H(\phi = \phi_0),$$

and simplify again using the overlap relations until we have a linear result in W :

$$\begin{aligned} H_c &= -2g^2 a \sum_{\sigma} \sum_{\sigma_1 < \sigma} \sum_{\sigma < \sigma_2} \int d\mathbf{q} W(\sigma_1, \sigma_2) \\ &\times \left((\sigma - \sigma_1)(\sigma_2 - \sigma_1)(\sigma_2 - \sigma) \left(2\lambda(\sigma) + (\mathbf{q}^2 + \mu^2) \frac{\sigma_2 - \sigma_1}{(\sigma_2 - \sigma)(\sigma - \sigma_1)} \right) \right)^{-1} \\ &- \sum_{\sigma} \lambda(\sigma) \rho(\sigma) + \frac{m_0^2}{2} \sum_{\sigma' > \sigma} \frac{W(\sigma, \sigma')}{\sigma' - \sigma}. \end{aligned} \quad (25)$$

Since the overlap relations are quadratic, once we have a linear expression in W , H_c cannot be further simplified.

In the above expression, the integral over \mathbf{q} is ultraviolet divergent at $D = 2$ and $D = 4$. This divergence can be eliminated by the mass renormalization and at $D = 4$ by also coupling constant renormalization. We observe that as $|\mathbf{q}| \rightarrow \infty$, the first term on the right, after doing the sum over σ , reaches a limit identical in form to the mass term. It can therefore be canceled by setting

$$m_0^2 = 4g^2a \int d\mathbf{q} \frac{1}{\mathbf{q}^2 + \mu^2}. \quad (26)$$

We note that without the use of the overlap relations, the structure of the ultraviolet divergence in (25) would be different from the structure of the mass counter term, and hence renormalization would not be possible. The presence of μ avoids an infrared divergence. At $D = 2$, there is no divergence, and at $D = 4$, a quadratic divergence is reduced to a logarithmic divergence in the coupling constant. Although there is no divergence at $D = 1$, we will still use the same expression for m_0 also in this case.

At the beginning, we started with two independent masses in the problem. But now that m_0 is fixed, only μ remains. We could have given a treatment based on a single mass from the start, however, having an extra mass temporarily is more convenient. For example, it enables us to give a uniform treatment for all dimensions.

Up to this point, apart from a few changes and amplifications, we have been reviewing parts of reference [5]. In the next section, we will finally break new ground. The starting point will be eq.(25). As it stands, there are two problems with this equation: The world sheet is still discrete, and the continuum limit $a \rightarrow 0$ is problematic. Also, there is the operator W , which has to be evaluated. Both problems will be addressed in the next section.

5. The Continuum Limit

The continuum limit is taken by letting $a \rightarrow 0$, after suitably scaling the field variables by

$$\phi \rightarrow \sqrt{a} \phi, \quad \psi \rightarrow \sqrt{a} \psi, \quad (27)$$

and similarly for the hermitian conjugate fields. From its definition, ρ scales as

$$\rho \rightarrow a \rho. \quad (28)$$

In the limit $a \rightarrow 0$, the scaled fields satisfy the commutation relations (2), with, however, the Kroenecker $\delta_{\sigma,\sigma'}$ replaced by the Dirac $\delta(\sigma,\sigma')$. From now on, to keep things simple, we will continue to use the same symbols ϕ and ψ also for the scaled fields, and whether the discretized or the continuum limit is being used should be clear from the context.

Let us now consider the continuum limit of H_c . In this limit, all the sigma sums become integrals, and all the factors of a are used up in this process. Also, the product in the definition of \mathcal{E} (eq.(5)) becomes

$$\mathcal{E}(\sigma_1, \sigma_2) = \prod_{\sigma_1}^{\sigma_2} (1 - a \rho(\sigma)) \rightarrow \exp \left(- \int_{\sigma_1}^{\sigma_2} d\sigma \rho(\sigma) \right). \quad (29)$$

After a change of variables by

$$\sigma = \sigma_1 + x (\sigma_2 - \sigma_1),$$

the result can be written as

$$\begin{aligned} H_c &= -2g^2 \int d\sigma_2 \int^{\sigma_2} d\sigma_1 \int_0^1 dx \int d\mathbf{q} \rho(\sigma_1) \mathcal{E}(\sigma_1, \sigma_2) \rho(\sigma_2) \\ &\times \left((\sigma_2 - \sigma_1) (2\lambda(\sigma) x(1-x)(\sigma_2 - \sigma_1) + (\mathbf{q}^2 + \mu^2)) \right)^{-1} \\ &+ 2g^2 \int d\sigma_2 \int^{\sigma_2} d\sigma_1 \int d\mathbf{q} \frac{1}{\mathbf{q}^2 + \mu^2} \frac{\rho(\sigma_1) \mathcal{E}(\sigma_1, \sigma_2) \rho(\sigma_2)}{\sigma_2 - \sigma_1} - \int d\sigma \lambda(\sigma) \rho(\sigma). \end{aligned} \quad (30)$$

The first and the second terms on the right are divergent as $|\mathbf{q}| \rightarrow \infty$ at $D = 2, 4$, and also they are also logarithmically divergent as $\sigma_2 - \sigma_1 \rightarrow 0$. The first is the ultraviolet mass divergence and we have already fixed m_0 by eq.(26) so that it cancels between the two terms. The second singularity is a logarithmic singularity at $\sigma_2 - \sigma_1 = 0$. Since $\sigma_2 - \sigma_1$ is the p^+ flowing through the propagator, this is the $p^+ = 0$ singularity in disguise. Surprisingly, this divergence also cancels between the first and second terms in all dimensions. It is highly satisfying that the mass counter term introduced to eliminate an ultraviolet divergence also automatically cancels the infrared divergence at $p^+ = 0$. This cancellation is quite non-trivial and absolutely essential, since otherwise, having only one adjustable constant m_0 at our disposal, we would be stuck with one divergence or other at $D = 2, 4$. We also note that we cannot add an arbitrary ultraviolet finite term to m_0^2 without spoiling the

infrared cancellation. Although we started with two masses, in the end only μ remains as an arbitrary parameter.

Another important feature of H_c is its symmetries. In addition to translation invariance in \mathbf{q} (eq.(17)), as is well known, the light cone dynamics is manifestly invariant under a subgroup of Lorentz transformations. The original action (16) is trivially invariant under all the generators of this subgroup except for the generator K_1 of boosts along the special direction 1. The discretization of the σ coordinate breaks this symmetry even at the classical level. We expect this symmetry will be at least classically restored in the continuum limit. To see this, we note that under K_1 , various fields transform as

$$\begin{aligned}\phi(\sigma, \tau, \mathbf{q}) &\rightarrow \sqrt{u} \phi(u\sigma, u\tau, \mathbf{q}), \quad \psi(\sigma, \tau,) \rightarrow \sqrt{u} \psi(u\sigma, u\tau), \\ \rho(\sigma, \tau) &\rightarrow u \rho(u\sigma, u\tau), \quad \lambda(\sigma, \tau) \rightarrow u \lambda(u\sigma, u\tau), \quad p^+ \rightarrow \frac{1}{u} p^+, \end{aligned} \quad (31)$$

where u parametrizes the K_1 transformations. In the expression for H_c , this amounts to letting

$$\sigma \rightarrow u \sigma, \quad \tau \rightarrow u \tau,$$

and transforming ρ according to eq.(31). The classical Hamiltonian then transforms as

$$H_c \rightarrow u H_c, \quad (32)$$

and as expected, the corresponding action is therefore invariant. As we shall see, this invariance will be respected by the mean field approximation, and it will play an important role in what follows.

Eq.(30), which is free of divergences and independent of a , will be the starting point of the mean field approximation in the next section. As we stressed earlier, making approximations at an earlier stage could have easily spoiled these desirable features.

6. The Meanfield Approximation

The mean field approximation consists of replacing ρ and λ in H_c by their ground state expectation values, which we assume to be independent of σ and τ . (translation invariance of the ground state). Afterwards, H_c should be minimized, subject to the constraints (3) and a (gauge) fixed λ , to find the ground state. Eq.(30) can then be simplified by the following change of variables:

$$\lambda = \lambda_0 \rho, \quad \sigma = \sigma' / \rho. \quad (33)$$

In terms of the new variables, we have,

$$\begin{aligned}
H_c = & p^+ \rho^2 \left(-\lambda_0 + 2g^2 \int_0^{\rho p^+} d\sigma' \int d\mathbf{q} \left(\frac{\exp(-\sigma')}{\sigma'} \right) \right. \\
& \times \left. \left(\frac{1}{\mathbf{q}^2 + \mu^2} - \frac{1}{\mathbf{q}^2 + \mu^2 + 2\lambda_0 x(1-x)\sigma'} \right) \right). \tag{34}
\end{aligned}$$

Let us now examine the structure of this equation. It can be written as

$$H_c = p^+ \rho^2 F(\lambda_0, \rho p^+) \tag{35}$$

The dependence of F on ρp^+ comes solely from the upper limit in the σ' integration. One could study H_c as a function of ρ and λ_0 without making any further approximations other than mean field, but we shall not attempt it in this paper. Instead, we will exclusively study the model in the small λ_0 limit. One motivation for this limit is that it is possible to get simple explicit results. Also, having already fixed $\lambda_0 p^+$ (see the discussion following eq.(7)), this corresponds to taking ρp^+ large. Since ρp^+ counts the total number of solid lines (propagators) on the world sheet, this corresponds to a world sheet densely covered with a large number of solid lines (propagators). This limit is of particular interest, since it is clearly non-perturbative and difficult to study by other methods. Also, it is in this limit that we may expect string formation on the world sheet.

From eq.(34), we see that in the large ρp^+ limit, we can let $\rho p^+ \rightarrow \infty$, and suppress the explicit dependence on ρp^+ in F . This amounts to neglecting exponentially suppressed boundary effects coming from the finite range of the σ' coordinate. Notice, however, that it is ρp^+ and not p^+ that is taken large. Letting p^+ become large is not K_1 invariant condition, since p^+ scales under K_1 . In contrast, $\lambda_0 p^+$ is K_1 invariant and taking it large is frame independent.

After dropping the dependence on the boundary effects, we can understand the structure of this equation as follows: The single factor of p^+ reflects the fact that on the world sheet that is uniform in σ , H_c is extensive in p^+ . The factor of ρ^2 is needed to have H_c transform correctly under K_1 (see eq.(31)). The function F could be arbitrary as far as K_1 invariance is concerned, since $\lambda_0 = \lambda/\rho$ is K_1 invariant. The general argument we have given shows that eq.(35) is valid in all transverse dimensions D and even after the addition of a ϕ^4 interaction.

We will determine F through the equation of motion with respect to λ , which, at fixed ρ , is the same as the equation of motion with respect to λ_0 :

$$\frac{\partial H_c}{\partial \lambda_0} = 0 \rightarrow \frac{\partial F}{\partial \lambda_0} = 0. \quad (36)$$

Since λ is fixed as an initial condition, this amounts to the determination of ρ in terms of λ . Carrying out the integration over \mathbf{q} in various dimensions, we have

$$F_D = -\lambda_0 + C_D g^2 \int_0^\infty d\sigma' \int_0^1 dx \frac{\exp(-\sigma')}{\sigma'} L_D(x, \sigma', \lambda_0), \quad (37)$$

where,

$$C_1 = 2\pi, \quad C_2 = 2\pi, \quad C_4 = 2\pi^2, \quad (38)$$

and,

$$\begin{aligned} L_1 &= \frac{1}{\mu} - \frac{1}{(\mu^2 + 2\lambda_0 x(1-x)\sigma')^{1/2}}, \\ L_2 &= \ln \left(1 + \frac{2\lambda_0 x(1-x)\sigma'}{\mu^2} \right), \\ L_4 &= 2\lambda_0 x(1-x)\sigma' \ln \left(\frac{\Lambda^2}{\mu^2} \right) \\ &\quad - \mu^2 \left(1 + \frac{2\lambda_0 x(1-x)\sigma'}{\mu^2} \right) \ln \left(1 + \frac{2\lambda_0 x(1-x)\sigma'}{\mu^2} \right). \end{aligned} \quad (39)$$

In the last equation, Λ is an ultraviolet cutoff. These equations fix the dimensionless parameter λ_0/μ^2 in terms of the dimensionless constants g^2/μ^3 at $D = 1$ and g^2/μ^2 at $D = 2$. At $D = 4$, the relation between g^2 and λ_0/μ^2 is more complicated and it is discussed below. We will explicitly evaluate these relations only in the small λ_0 limit by expanding to first order in λ_0 (see the discussion following eq.(35)):

$$\begin{aligned} \frac{\partial F_1}{\partial \lambda_0} &= 0 \rightarrow \frac{\lambda_0}{\mu^2} = \frac{5}{3} - \frac{5\mu^3}{\pi g^2} + \dots, \\ \frac{\partial F_2}{\partial \lambda_0} &= 0 \rightarrow \frac{\lambda_0}{\mu^2} = \frac{5}{2} - \frac{15\mu^2}{4\pi g^2} + \dots, \\ \frac{\partial F_4}{\partial \lambda_0} &= 0 \rightarrow \frac{1}{g^2} = \pi^2 \left(\frac{2}{3} (\ln(\Lambda^2/\mu^2) - 1) - \frac{4\lambda_0}{15\mu^2} + \dots \right). \end{aligned} \quad (40)$$

The first two equations give us conditions on the coupling constant. Since the left hand sides of these equations are positive, it follows that

$$\frac{g^2}{\mu^3} \geq \frac{3}{\pi}$$

at $D = 1$ and,

$$\frac{g^2}{\mu^2} \geq \frac{3}{2\pi}$$

at $D = 2$. Otherwise, there is no solution to these equations. At least in this approximation, these inequalities are then the conditions for the formation of a world sheet densely populated with solid lines. We note that expanding in λ_0 , is the same as expanding in the small parameter

$$1 - \frac{3\mu^3}{\pi g^2}$$

at $D = 1$ and in

$$1 - \frac{3\mu^2}{2\pi g^2}$$

at $D = 2$. Although we will not pursue it here, one can envisage a systematic expansion in these small parameters.

In the case of $D = 4$, λ_0 need not really be small,

$$\Lambda^2/\mu^2 \gg 1,$$

is all that is needed, as discussed below. In this case, the expansion parameter is the running coupling constant.

The last equation, at $D = 4$, has a logarithmic dependence on the cutoff Λ . This is related to the coupling constant renormalization. We recall that ϕ^3 is asymptotically free in 6 space-time dimensions ($D = 4$), and the above relation is the well known lowest order renormalization group result obtained by summing the leading logarithmic divergences in the perturbation series. To get a finite result, one should first renormalize the coupling constant before summing the logs. This amounts to replacing the cutoff Λ by a large but finite value. The coupling constant on the left should then be identified with the running coupling constant $g(\Lambda)$, defined at the energy scale Λ . For this leading log. approximation to be reliable, $g(\Lambda)$ should be small, which means that Λ^2/μ^2 should be large. All the additional terms on the right hand

side, including the term proportional to λ_0 , only make a small change in the scale of the running coupling constant.

An interesting quantity is the ground state energy. It can be computed to lowest order in λ from equations (35), (37) and (40):

$$\begin{aligned}
H_c(D=1) &= \frac{3}{10} \frac{p^+ \lambda^2}{\mu^2} + \dots = \frac{3}{10} \frac{\mu^2 \bar{\lambda}^2}{p^+} + \dots, \\
H_c(D=2) &= \frac{3}{10} \frac{p^+ \lambda^2}{\mu^2} + \dots = \frac{3}{10} \frac{\mu^2 \bar{\lambda}^2}{p^+} + \dots, \\
H_c(D=4) &= \frac{1}{5 (\ln(\Lambda^2/\mu^2) - 1)} \frac{p^+ \lambda^2}{\mu^2} + \dots = \frac{1}{5 (\ln(\Lambda^2/\mu^2) - 1)} \frac{\mu^2 \bar{\lambda}^2}{p^+} + \dots,
\end{aligned} \tag{41}$$

where,

$$\bar{\lambda} = \frac{p^+ \lambda}{\mu^2}$$

is a pure number; it is dimensionless and invariant under K_1 (see eq.(31)). The light cone energy is p^- , so multiplying it by p^+ , we find that the invariant mass squared of the ground state is proportional to $\bar{\lambda}^2 \mu^2$. Notice that the mass squares are all positive. At least, in this approximation, there is no instability.

7. Singularity Cancellation: $\phi^3 + \phi^4$ In 1+2 Dimensions

In this section, we add a ϕ^4 interaction to the ϕ^3 of the previous section, and show that, just as in the case of pure ϕ^3 , the singularity at $p^+ = 0$ can be canceled by a mass counter term. The idea is to demonstrate that the elimination of this singularity is not a specific feature of the ϕ^3 interaction but it can be generalized to include an additional ϕ^4 term. We will consider the model only at $D = 1$, and thereby avoid the complications resulting from the renormalization of the ϕ^4 coupling constant at $D = 2$. These complications can be handled just as in the case of ϕ^3 at $D = 4$, but we will skip it in the interests of simplicity and brevity. We will also not study the model in any detail, beyond showing the elimination of the singularity.

The total interaction Hamiltonian is with this addition is given by

$$H_{I,t} = H_I + H'_I, \tag{42}$$

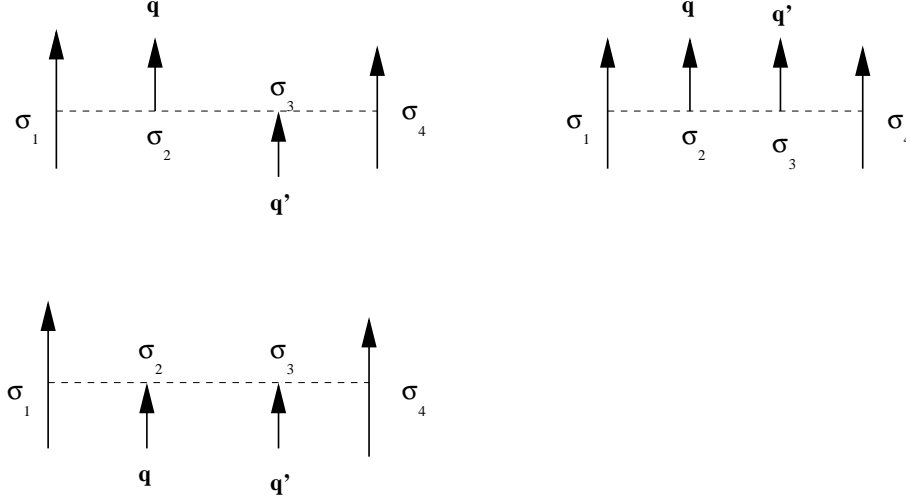


Figure 4: Four Point Vertices

where H_I is given by eq.(11) and,

$$\begin{aligned}
 H'_I &= g' a \sum_{\sigma_{1,2,3,4}} \int d\mathbf{q} \int d\mathbf{q}' \phi^\dagger(\sigma_2, \mathbf{q}) \rho_-(\sigma_2) \mathcal{E}(\sigma_1, \sigma_4) \rho_+(\sigma_3) \phi(\sigma_3, \mathbf{q}') \\
 &\times ((\sigma_2 - \sigma_1)(\sigma_4 - \sigma_2)(\sigma_4 - \sigma_3)(\sigma_3 - \sigma_1))^{-1/2} + \dots .
 \end{aligned} \tag{43}$$

Here, g' is a positive coupling constant, scaled by a in order that in the limit $a \rightarrow 0$, sums over σ smoothly go over integrals over σ (see section 3). The sum over the σ 's is non-zero only for

$$\sigma_1 < \sigma_2 < \sigma_4, \quad \sigma_1 < \sigma_3 < \sigma_4.$$

Actually, the term we have written out in the expression for H'_I corresponds to first graph in Fig.4 . We will demonstrate the elimination of the singularity only for this graph; the other two graphs can be treated along the same lines.

We are now ready to write down the equations of motion, similar to (22), for this extended model. Actually, the algebra can be greatly simplified by setting $\lambda = 0$ at the beginning. This is because singularity cancellation occurs at $\lambda = 0$, or equivalently, at $\lambda_0 = 0$. In the case of ϕ^3 interaction, this can be seen from eq.(25): Expanding the integral on the right hand side in powers of $\lambda_0 = 0$, we notice that the zeroth order terms cancel, and the

higher order terms depend only on the powers of $\lambda_0 = 0$ in the combination $\lambda_0 \sigma'$. Therefore, the potential singularity at $\sigma' = 0$ is absent because of the extra factors of σ' . This is true even after the addition of a ϕ^4 interaction; higher order terms in λ_0 always come with extra powers of σ' and therefore they are singularity free.

The equation of motion for $\phi_0(\sigma)$,

$$\begin{aligned} \sum_{\sigma'} G(\sigma, \sigma') \rho(\sigma') (\mathbf{q}^2 + \mu^2) \phi_0(\sigma, \mathbf{q}) &= 2g\sqrt{a} \rho_-(\sigma) \mathcal{V}(\sigma) \\ + g'a \int d\mathbf{q}' \sum_{\sigma_1 < \sigma' < \sigma_2} &\frac{\rho_-(\sigma) W(\sigma_1, \sigma_2) \phi_0(\sigma', \mathbf{q}')}{((\sigma_2 - \sigma)(\sigma - \sigma_1)(\sigma_2 - \sigma')(\sigma' - \sigma_1))^{1/2}}, \end{aligned} \quad (44)$$

differs from eq.(22) in two respects: λ is set equal to zero and there is an extra term on the right hand side coming from the ϕ^4 interaction. It turns out that this equation is solved by the ansatz

$$\phi_0(\sigma, \mathbf{q}) = 2A\sqrt{a} \sum_{\sigma_1 < \sigma < \sigma_2} \frac{\rho_-(\sigma) W(\sigma_1, \sigma_2) ((\sigma - \sigma_1)(\sigma_2 - \sigma))^{1/2}}{(\mathbf{q}^2 + \mu^2) (\sigma_2 - \sigma_1)^{3/2}}. \quad (45)$$

This ansatz is almost the same as eq.(23), with the only difference that $\lambda = 0$ and the overall coefficient A is left arbitrary. Substituting in the equation of motion (44) and simplifying by the use of the overlap relations, we find that if

$$A = - \left(g + \frac{2\pi}{\mu} A g' \right), \quad (46)$$

the equation of motion is satisfied. The resulting H_c has the same form as eq.(34), only with $\lambda_0 = 0$ and g^2 replaced by A^2 . The mass counter term m_0^2 is given by eq.(26) as before, with again g^2 replaced by A^2 . With these changes, just as in the case of pure ϕ^3 , the result is still non-singular at $\sigma' = 0$. Therefore, even with the addition of a ϕ^4 interaction, one can go to the continuum limit without encountering any singularities.

8. Fluctuations Around The Classical Background

Given the classical solutions developed in the previous sections, it is natural to study quantum fluctuations about these backgrounds. This can be done explicitly to quadratic order for all the fluctuations. We will, instead, focus exclusively on a particular set of fluctuations, which were studied also in

the earlier work . These are obtained by quantizing the collective coordinates corresponding to broken translation invariance

$$\mathbf{q} \rightarrow \mathbf{q} + \mathbf{r}. \quad (47)$$

(See eq.(17)). The classical solution, placed at a definite location in the \mathbf{q} space, breaks this symmetry. This is familiar from solitons in field theory, and the symmetry is restored by quantizing the so-called collective modes. These modes are very important not only for their role in restoring translation invariance, but also, because, they are the low lying Goldstone modes connected with the spontaneously broken translation symmetry. In the earlier work, they were crucial to the formation of a string on the world sheet. Also, as we shall see, their contribution to the action can be computed exactly. For all these reasons, we will study only these collective modes and not consider any other modes in this work.

Inevitably, there will be some overlap with the earlier work [5]. However, we are not here trying merely to reproduce previous results; in fact, with our new approach, we arrive at a somewhat different picture, such as no-linear string trajectories, which we will explain shortly.

The collective coordinate corresponding to translations is introduced by letting

$$\phi = \phi_0 + \phi_1, \quad (48)$$

where ϕ_1 is the fluctuating part of the field, and setting,

$$\phi_1(\sigma, \tau, \mathbf{q}) = \phi_0(\sigma, \mathbf{q} + \mathbf{v}(\sigma, \tau)) - \phi_0(\sigma, \mathbf{q}), \quad (49)$$

where ϕ_0 is the classical solution and \mathbf{v} is the collective coordinate. The contribution of ϕ_1 to the action can be written as the sum of kinetic and potential terms:

$$S^{(1)} = S_{k.e} - \int d\tau H_0(\phi_1) = S_{k.e} + S_{p.e}, \quad (50)$$

where the kinetic term depends on $\partial_\tau \mathbf{v}$ and the potential has no τ derivatives. We note that only H_0 contributes to $S_{p.e}$: There are no τ derivatives and the linear ϕ terms in H_I have been eliminated by shifting ϕ by the classical solution.

We now substitute the ansatz (49) directly into H_0 (eq.(8)). The integrals over \mathbf{q} and \mathbf{q}' can be done by shifting them by \mathbf{v} , and the result can be

simplified by invoking the constraint (3) and the rotation invariance of ϕ_0 , with the result:

$$S_{p.e} = -\frac{1}{4} \int d\tau \int d\sigma \int d\sigma' \frac{W(\sigma, \sigma')}{|\sigma - \sigma'|} (\mathbf{v}(\sigma, \tau) - \mathbf{v}(\sigma', \tau))^2. \quad (51)$$

We note that so far no approximation was made, and therefore, this result is exact so long as only the contribution of the collective coordinate \mathbf{v} is concerned. Also, there is no singularity at $\sigma = \sigma'$ and so there is no obstacle to taking the continuum limit immediately. To make further progress, we introduce the mean field approximation by setting ρ to be a constant, with the result

$$S_{p.e} \rightarrow -\frac{\rho^2}{4} \int d\tau \int d\sigma \int d\sigma' \frac{\exp(-\rho |\sigma - \sigma'|)}{|\sigma - \sigma'|} (\mathbf{v}(\sigma, \tau) - \mathbf{v}(\sigma', \tau))^2. \quad (52)$$

We will study this action in detail later on, but before that, we turn our attention to the kinetic energy term. We will first compute this term to quadratic order in $\partial_\tau \mathbf{v}$, and later argue that the result is exact. Since the action (16) is first order in the τ derivative, to cast it into a quadratic form in $\partial_\tau \phi_1$, one has to split ϕ_1 into its real and imaginary (Hermitian and anti-Hermitian) parts:

$$\phi_1 = \phi_{1,r} + \phi_{1,i}, \quad (53)$$

and eliminate one of them by integrating over it. In this case, since the classical solution ϕ_0 is real, the ansatz (49) more precisely applies only to the real part of ϕ_1 :

$$\phi_{1,r}(\sigma, \tau, \mathbf{q}) \rightarrow \phi_0(\sigma, \mathbf{q} + \mathbf{v}(\sigma, \tau)), \quad (54)$$

and $\phi_{1,i}$ will be integrated out. The kinetic energy term in the action (16) can then be rewritten as

$$\begin{aligned} i \sum_\sigma \int d\tau \int d\mathbf{q} \phi^\dagger \partial_\tau \phi &= 2 \sum_\sigma \int d\tau \int d\mathbf{q} \phi_{1,i} \partial_\tau \phi_{1,r} \\ &\rightarrow 2 \sum_\sigma \int d\tau \int d\mathbf{q} \phi_{1,i} \partial_\tau \phi_0(\sigma, \mathbf{q} + \mathbf{v}(\sigma, \tau)). \end{aligned} \quad (55)$$

Now consider the contribution coming from H_0 . We have already taken care of the contribution of $\phi_{1,r}$ when we computed $S_{p.e}$, so it remains to compute the quadratic terms in $\phi_{1,r}$. Integrating over $\phi_{1,i}$ then amounts to solving the equations of motion for $\phi_{1,i}$ and substituting in the action. The

left hand side of the equation of motion is the same as in (22), but the right hand side comes from the variation of the above kinetic term with respect to $\phi_{1,i}$:

$$\left(2\lambda(\sigma) + \sum_{\sigma'} G(\sigma, \sigma') \rho(\sigma') (\mathbf{q}^2 + \mu^2) \right) \phi_{1,i}(\sigma, \tau, \mathbf{q}) = 2\partial_\tau \phi_0(\sigma, \mathbf{q} + \mathbf{v}(\sigma, \tau)). \quad (56)$$

This equation can be solved by letting

$$\phi_{1,i}(\sigma, \tau, \mathbf{q}) = \sum_{\sigma_1 < \sigma} \sum_{\sigma < \sigma_2} \rho_-(\sigma) W(\sigma_1, \sigma_2) \tilde{\phi}_{1,i}(\sigma, \sigma_1, \sigma_2, \tau, \mathbf{q}), \quad (57)$$

as in (23). Following the same steps as before, this can then be simplified using the overlap relations, and after some algebra, we have the solution

$$\tilde{\phi}_{1,i}(\sigma, \sigma_1, \sigma_2, \tau) = \frac{2 \partial_\tau \mathbf{v}(\sigma, \tau) \cdot \nabla_q \tilde{\phi}_0(\sigma, \sigma_1, \sigma_2, \tau, \mathbf{q})}{\left(2 \lambda(\sigma) + (\mathbf{q}^2 + \mu^2) \left(\frac{\sigma_2 - \sigma_1}{(\sigma_2 - \sigma)(\sigma - \sigma_1)} \right) \right)}, \quad (58)$$

where $\tilde{\phi}_0$ is given by (24). Substituting this back in the action, and after some more use of the overlap relations, we arrive at a result free of singularities. We can then take the continuum limit, and apply the mean field approximation. We skip the intermediate steps of the straightforward algebra and give the final result:

$$S_{k.e} \rightarrow \int d\tau \int d\sigma \frac{1}{2} E(\lambda_0) (\partial_\tau \mathbf{v}(\sigma, \tau))^2, \quad (59)$$

where,

$$E(\lambda_0) = \int_0^\infty d\sigma_1 \int_0^\infty d\sigma_2 \int d\mathbf{q} \frac{64 g^2 \mathbf{q}^2}{D} \frac{\sigma_1^2 \sigma_2^2 (\sigma_1 + \sigma_2) \exp(-(\sigma_1 + \sigma_2))}{(2 \lambda_0 \sigma_1 \sigma_2 + (\mathbf{q}^2 + \mu^2) (\sigma_1 + \sigma_2))^5}. \quad (60)$$

A few comments about this result are in order:

- a) The integrals in the expression for E are convergent in the dimensions we are considering, and therefore, there are no problems with singularities.
- b) E is independent of σ . This follows quite generally from translation invariance in the σ coordinate in the mean field approximation. As a consequence, $S_{k.e}$ is local both in σ and τ . In contrast, $S_{p.e}$ is non-local in σ (eq.(52)). Since λ_0 is already determined in terms of g and μ (see eq.(40)), E is just a fixed normalization constant.

From the derivation given above, it may seem that we have only taken into account the quadratic terms in ϕ_1 . However, we shall now argue that the result for $S_{k.e}$ is more general, and does not depend on this approximation. This follows from the following properties of $S_{k.e}$: It is local, it is rotation invariant in \mathbf{q} , and it invariant under

$$\mathbf{v}(\sigma, \tau) \rightarrow \mathbf{v}(\sigma, \tau) + \mathbf{r}, \quad (61)$$

\mathbf{r} is a constant vector. This follows from the translation invariance of \mathbf{q} (eq.(17)). A general term local, second order in ∂_τ , and rotation invariant has to be of the form

$$\Delta S = \int d\tau \int d\sigma \int d\mathbf{q} (\partial_\tau \mathbf{v}(\sigma, \tau))^2 Z((\mathbf{v}(\sigma, \tau))^2),$$

but this is not invariant under the translation of \mathbf{v} by \mathbf{r} unless the function Z is a constant. The remaining possibility is a term fourth order in ∂_τ , of the form

$$((\partial_\tau \mathbf{v}(\sigma, \tau))^2)^2,$$

coming from the fourth order term

$$-\frac{1}{2} \sum_{\sigma, \sigma'} G(\sigma, \sigma') \int d\mathbf{q} \int d\mathbf{q}' (\mathbf{q} \cdot \mathbf{q}') \phi^\dagger \phi(\sigma, \mathbf{q}) \phi^\dagger \phi(\sigma', \mathbf{q}')$$

in H_0 , but rotation invariance in \mathbf{q} forbids such a term. Of course, this simple result holds only for the contribution coming from \mathbf{v} ; all sorts of other quantum fluctuations have been neglected.

9. String Formation

In this section, we will study the spectrum of the collective coordinate \mathbf{v} , with the action given by the sum of $S_{p.e}$ (eq.(52)) and $S_{k.e}$ (eq.(59)). This is a free field theory and therefore it is exactly solvable. The kinetic energy is already diagonal, and the potential term can be diagonalized by defining

$$\mathbf{v}(\sigma) = \frac{1}{2\pi} \int dl e^{-il\sigma} \tilde{v}(l).$$

In terms of \tilde{v} , the Hamiltonian $H_{p.e}$ corresponding to $S_{p.e}$ becomes diagonal:

$$H_{p.e} = -\frac{\rho^2}{4\pi} \int dl \tilde{v}(l) \cdot \tilde{v}(-l) \ln \left(\frac{\rho^2}{\rho^2 + l^2} \right). \quad (62)$$

So far, we have been treating l as a continuous variable, but since σ is compactified on circle of circumference p^+ , l should be discretized:

$$l \rightarrow \frac{2\pi n}{p^+}.$$

The integral over l is then replaced by

$$\int dl \rightarrow \frac{2\pi}{p^+} \sum_l.$$

Now consider the limit of large ρ , which corresponds to $\lambda_0 \rightarrow 0$. More precisely, we consider the range $l \ll \rho$. so that

$$\ln \left(\frac{\rho^2}{\rho^2 + l^2} \right) \rightarrow -\frac{l^2}{\rho^2},$$

to the lowest order in l , and,

$$H_{p.e} \rightarrow \frac{2}{p^+} \sum_l l^2 \tilde{v}(l) \cdot \tilde{v}(-l) = \frac{1}{2} \int_0^{p^+} d\sigma (\partial_\sigma \mathbf{v}(\sigma))^2. \quad (63)$$

In this limit, the action for \mathbf{v} tends to the string action:

$$S \rightarrow \frac{1}{2} \int d\tau \int_0^{p^+} d\sigma (E (\partial_\tau \mathbf{v}(\sigma, \tau))^2 - (\partial_\sigma \mathbf{v}(\sigma, \tau))^2). \quad (64)$$

However, this is true only in extreme limit $\rho \rightarrow \infty$. For a finite ρ , no matter how large, the Regge trajectories are no longer linear, and therefore, the string picture is only approximately valid for relatively low lying states. Noting that we are dealing with a scalar field theory, this deviation from the strict string picture should not be surprising.

10. Conclusions

As we have already mentioned in the introduction, we expect the present work to conclude a long series of investigations of scalar field theory on the light cone world sheet. We have here used the methods and tools developed in especially reference [5]. Where we differ from this reference is in the handling of the continuum limit $a \rightarrow 0$. We show that with an appropriate choice of the mass counter term, both the singularity at $a = 0$ and the ultraviolet

mass divergences are eliminated. This is an important advance; we have finally a result free of cutoff, which made the previous work unreliable. The results about the ground state of the model and string formation do not differ drastically from the earlier results, but they are now much more trustworthy.

There remains still the problem of manifest Lorentz invariance. Within the framework of the approach used here, an initial step in this direction has been taken [13]. It remains to see how far one can push it as a practical method. Also, elimination of the cutoff dependence makes it easier to generalize to more interesting theories, such as gauge theories. One of the simplest of these, non-Abelian theory in 1+2 dimensions, seems to be almost within reach.

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